VOLUME 76 SEPARATE No. 18



# PROCEEDINGS

## AMERICAN SOCIETY OF CIVIL ENGINEERS

MAY, 1950

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

# SUCCESSIVE APPROXIMATIONS FOR BEAMS ON AN ELASTIC FOUNDATION

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Headquarters of the Society
33 W. 39th St.
New York 18, N.Y.



PRICE $0.50 PER COPY

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AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
PAPERS

# SUCCESSIVE APPROXIMATIONS FOR BEAMS ON AN ELASTIC FOUNDATION

BY E. P. POPOV,[1] ASSOC. M. ASCE

SYNPOSIS

A method resembling the well known moment-area procedure is presented for the solution of beams on an elastic foundation. Essentially it is an extension, adaptation, and systematization of the famed Vianello-Stodola procedures as are very briefly mentioned by August Föppl[2] and Miklós Hetényi.[3] Because of the inherent power of these procedures, reasonably accurate results are obtained simply and quickly. The difficult mathematics that occur in the usual analytical solutions are completely avoided.

## INTRODUCTION

The supporting media, such as soil, as well as a load-carrying beam or footing that rests on it, usually display a certain amount of elastic behavior; yet, for simplicity, it is customary in most cases to consider the footing to be infinitely rigid. The design is based on the assumption of uniform soil pressure distribution if the centroid of the footing coincides with the resultant of the applied forces. Similar simplified treatment is used for eccentric loads. Such an assumption becomes less and less accurate as the footing or the load-carrying beam becomes relatively more flexible. For example, in the case of a railroad rail supported by ties, the ordinary solution would lead to erroneous results and cannot be used. There is an important class of problems in which flexibility of the beam on an elastic foundation must be considered.

If the supporting medium is considered as a true elastic continuum, the problem becomes unusually difficult and only a few[4] rigorous mathematical solutions exist. However, if the reaction forces are assumed to be proportional

NOTE.—Written comments are invited for publication; the last discussion should be submitted by October 1, 1950.

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[2] "Vorlesungen über Technische Mechanik," by August Föppl, B. G. Teubner, Leipzig, 1900, Vol. III, p. 264.

[3] "Beams on Elastic Foundation," by Miklós Hetényi, The Univ. of Michigan Press, Ann Arbor, Mich., 1946, p. 99.

[4] *Ibid.*, pp. 197–214.

to the deflection of the beam at that point, the problem becomes greatly simplified; yet, the solution of the simplified problem appears to yield results that are in reasonable agreement with the existing exact ones. In the case of a foundation problem, rather complex soil phenomena are involved and an extreme mathematical refinement in the solution of a practical problem seems unwarranted. Moreover, in many problems, such as those occurring at the stiffening rings in thin shells, grillage beams, etc., the foregoing assumption holds rigorously.

The idealized foundation assumed here may be thought of as one composed of an infinite number of independent springs. The elastic constant of these "springs" represents the foundation modulus. These springs can take both tension and compression forces. This assumption was first introduced[5] by Eduard Winkler. Many subsequent investigators used this assumption, perfected the mathematical aspect of the problem, and, in general, found good agreement between the experimental and analytical results. The earlier works of Hermann Zimmermann[6] and Keiichi Hayashi[7] are notable. A method developed by A. N. Krilov[8] is rather convenient and has been introduced into a textbook.[9,10] A book by Mr. Hetényi[3,11] discusses the same problem in a most complete and thorough manner. In this work, in addition to tables and complete solutions of many special cases, some series solutions are used which greatly facilitate the utility of the method for practical problems.

However, the foregoing procedures are thought by some to be difficult as they involve lengthy mathematical expressions. To overcome this difficulty a method by Zusse Levinton,[12] M. ASCE, was developed in which the entire procedure is mainly reduced to the solution of simultaneous equations. The determination of coefficients and constants in many cases may be routinized. This method involves some legitimate approximations from the engineering point of view. It becomes quite cumbersome, however, as the complexity of the problem increases.

To avoid the tediousness of a mathematical solution, ingenious electrical arrangements have been developed by A. L. Goflin[13] for solving the necessary differential equations. The method appears to be rapid, it can provide for the variable moment of inertia of the beam, but it requires special equipment and techniques.

To overcome some of the difficulties mentioned in this brief review of the available methods an approximate procedure is developed in this paper. The

[5] "Die Lehre von der Elastizitaet und Festigkeit mit Besonderer Rücksicht auf ihre Anwendung in der Technik," by Eduard Winkler, Dominicus, Prag, 1867, p. 182.

[6] "Die Berechnung des Eisenbahnoberbaues," by Hermann Zimmermann, W. Ernst and Sohn, Berlin, 2d Ed., 1930.

[7] "Theorie des Trägers auf Elastischer Unterlage und ihre Anwendung auf den Tiefbau," by Keiichi Hayashi, Julius Springer, Berlin, 1921.

[8] "Analysis of Beams on Elastic Foundations," by A. N. Krilov, Akademyia Nauk, Leningrad, 2d Ed., 1931.

[9] "Strength of Materials," by M. M. Filonenko-Borodich, *Stroiisdat*, Moscow, 1940, pp. 315–337.

[10] *Ibid*, pp. 540–551.

[11] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., Inc., New York, N. Y., 2d Ed., 1941, Pt. II, p. 1.

[12] "Elastic Foundations Analyzed by the Method of Redundant Reactions," by Zusse Levinton, *Transactions*, ASCE, Vol. 114, 1949, pp. 40–52.

[13] "An Electric Model of an Elastically Supported Beam," by A. L. Goflin, *Bulletin*, Akademyia Nauk, Moscow, December, 1946, pp. 1743–1751.

method proposed utilizes the great power of graphical integration. In many practical problems the moment of inertia of a beam as well as the foundation modulus are variable quantities along the length of the beam. Such variation may be discontinuous. Even the original mathematical formulation for the moment of inertia and the foundation modulus may require some approximations. Thereafter the solution may become very complex or entirely untractable. Although it lacks extreme accuracy, the method proposed overcomes these complexities. Essentially the problems may be solved with nearly the same ease whether they are simple or complex in the sense of mathematical formulation. The steps used in the solution of the problem are all very simple and are well known to engineers.

## General Procedure

The usual assumption of a rigid beam on an elastic foundation is shown in Fig. 1(*a*), the true situation for a flexible beam is shown in Fig. 1(*b*), and in an idealized elastic foundation only the "springs" directly under the beam are

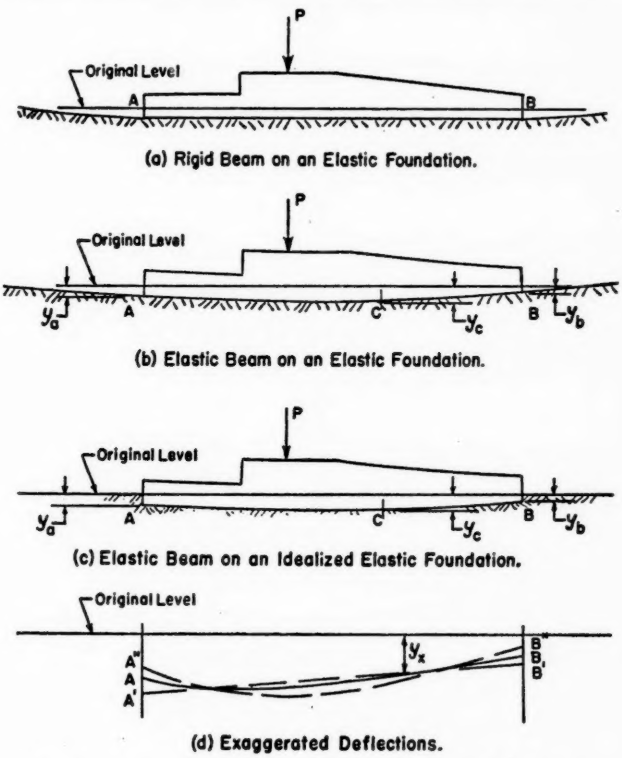


Fig. 1.—General Case of a Beam on an Elastic Foundation

loaded and deflect as shown in Fig. 1(*c*). The beam, being flexible, assumes some such curve as AB shown exaggerated in Fig. 1(*d*). Since the foundation is elastic, at point A the force exerted by the spring on the beam is $k_a\, y_a$, in which

$k_a$ is the foundation modulus at point A and $y_a$ is the deflection of the beam at point A. Similarly, at point B such a force is $k_b\, y_b$, and at point C it is $k_c\, y_c$. Thus, knowing the foundation moduli, the soil pressure diagram can be easily constructed in a manner corresponding to a deflected beam. Statics considerations require that the resultant of the foundation pressure be equal and opposite to the forces applied to the beam, including its own weight. In this method this requirement must be kept in mind constantly as it is used over and over again to adjust the solutions obtained.

Initially the shape of the elastic beam is not known and must first be assumed perfectly rigid. This is the very assumption used in ordinary design and its accuracy for the final solution has been questioned herein. Under this assumption a symmetrically loaded beam will settle an equal amount everywhere if the foundation constant is uniform or variable, and symmetrical in relation to the beam. If the loading, or the foundation modulus, is unsymmetrical, the "rigid" beam will rotate as well as settle. The amount of this angular movement can be computed easily. Thus (thinking for a moment in terms of stresses, which can be converted into deflections once the foundation modulus is known), when there is rotation, the stresses are $s = P/A \pm M\, c/I$ instead of $s = P/A$ of the former case. The symbols in these relations have the usual meaning: $P$ is the resultant of the applied loads; $A$ is simply the area under the footing if $k$ is uniform, or the transformed area (see Example 3, subsequently) if $k$ is variable; $c$ is the distance from the neutral axis to the part of the foundation investigated; $I$ is the moment of inertia of the footing contact area (transformed area if $k$ is variable) about its neutral axis; and $M$ is the "rotating" moment, which is equal to $P$ multiplied by the distance from the centroid of the footing area. In this manner, regardless of the type of loading or the foundation moduli, the pressure distribution under a rigid beam may be determined; and, knowing the pressure distribution, the corresponding deflections can be determined if desired.

The result of such calculations could be represented by a straight line A′ B′ in Fig. 1(*d*), and the corresponding pressure distribution under the same beam would be $k_x\, y_x$. The subscript $x$ in this relation designates the particular modulus and the particular deflection at a distance $x$ from the end of the beam. Then the beam is considered flexible and its elastic curve is determined under the influence of the actual applied loads and the $k_x\, y_x$-loading. This is done by plotting the usual $\dfrac{M}{E\,I}$-diagram, subdividing it into a convenient number of segments, and graphically or arithmetically determining the elastic curve by the conjugate beam method. (In the expression $M/(E\,I)$, the symbol $M$ denotes a bending moment at a section, and $I$ is the moment of inertia of the corresponding cross-sectional area.) These deflections variously increase and relieve the foundation pressures, upsetting the static balance. By "locking" the beam in the curved shape just determined the results are adjusted to bring them into static balance—that is, the resultant of the foundation pressure under the deflected beam is made to be the equilibrant of the applied forces. This procedure is similar to the original treatment given a rigid beam, except that a constant force caused by the deflection of the beam enters into the calculations.

Such results are again shown diagrammatically in Fig. 1(*d*) by the curved line A″ B″. The shape of the curve A″B″ is the elastic curve previously determined by the conjugate beam method.

Curve A″ B″ exaggerates the deflection of the elastic curve as it was based on the loading that would occur on an infinitely rigid beam. In the case illustrated, the ends of the beam actually would "escape" such heavy loading by deflecting as a result of the flexibility of the beam. These deflections, and hence the corresponding pressures, "overshoot" the true values. The correct answer lies somewhere between the line A′ B′ and the curve A″ B″. Usually the best estimate that can be made at the end of this cycle is that the true elastic curve is the average between A′B′ and A″B″. For the more rigid members the answer is closer to A″B″, but for flexible members, it is closer to A′B′. In the latter case, if A″B″ is taken as the approximation for subsequent work, the solution may not converge. Thus for uniformity and to avoid the danger of working with a nonconvergent solution, it is recommended that the average of A′B′ and A″B″ be used.

After averaging the ordinates of curves A′B′ and A″B″, the first cycle of the solution is complete. The corresponding foundation pressures are $k_x\, y$, in which $y$ is the average ordinate of the two curves at a point $x$. In order to improve the accuracy of the solution, the pressure distribution corresponding to the foregoing average deflection is then used. This pressure distribution, together with the actual loads, is used to obtain a new $\frac{M}{E\,I}$ -diagram, and, proceeding as before, a new elastic curve is obtained. Then, it is adjusted for static compatibility. A beam "locked" into the curved shape of the new elastic curve is used in such calculations. The average of the initial elastic curve of this cycle with the new adjusted elastic curve is the desired answer for the two cycles.

It has been found that a solution made on the foregoing basis converges very rapidly. Seldom are more than two cycles required.

To illustrate the details of this method three examples follow, in which some pertinent points are discussed. Cases of variable moment of inertia, variable foundation moduli, and the utility of the transformed areas for a given case are shown. To facilitate comparison with other methods, data for Examples 2 and 3 are from published works to be cited. However, the type of problem in Example 1, which is unusually simple to solve by the method discussed, is normally avoided, as it presents some mathematical difficulties.

## Example 1

Consider a continuous concrete footing having a cross section (Fig. 2(*a*)) with a line load of 150 kips per ft (k/ft). The elastic modulus of concrete is taken as $E = 432{,}000$ kips per sq ft, and the foundation modulus is assumed to be $k_o = 300$ kips per cu ft. In this example the soil pressure distribution is found for the foregoing conditions. The weight of the beam is neglected.

Assuming the footing to be rigid, the pressure distribution is $s = 150/30 = 5$ kips per sq ft (k/ft²), and the uniform deflection is $\Delta = 5/300 = 0.0167$ ft. Considering a strip of the footing of unit width, the bending moment diagram is

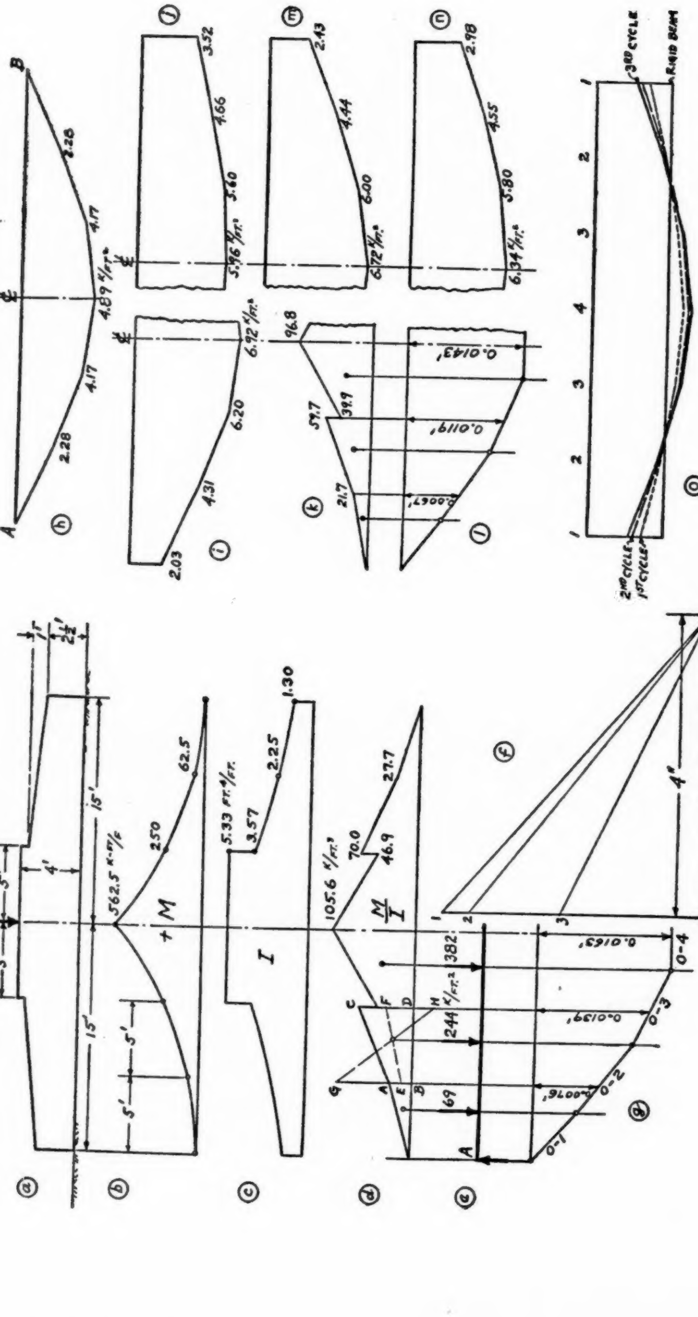

FIG. 2.—EXAMPLE 1, SYMMETRICAL LOADING

computed for values of 5 kips per ft up, and 150 kips per ft down (Fig. 2(*b*)). In Fig. 2(*c*), a plot of the moment of inertia of the cross-sectional area is shown; this was computed at 5-ft intervals using the relation $I = \frac{1}{12} b\ d^3$. For example, at the center $I = \frac{1}{12} \times 1 \times 4^3 = 5.33$ ft$^4$ per ft, and directly below this, in Fig. 2(*d*), the $\frac{M}{I}$-diagram is shown. Only four values are computed and the actual curve is approximated by a series of straight lines. Only one half of the diagram is shown as this problem deals with complete symmetry. The $M/I$-diagram gives the elastic loading for the conjugate beam. The distributed loadings are concentrated at their respective centers of gravity. It will be recalled that such centers for triangles are located at a distance of one third from the base. From a trapezoid, one way of finding the center of gravity is to bisect distances such as AB and CD (Fig. 2(*d*)) and also to lay them off on the opposite sides as shown. This procedure yields points E, F, H, and G. The intersection of lines GH and EF gives the desired point.

To compute the elastic weights on the conjugate beam (Fig. 2(*e*)) simply add the areas of the $\frac{M}{I}$-diagram for a particular segment. For example, $\frac{1}{2} \times 5 \times (27.7 + 70.0) = 244$ kips per sq ft. Then the elastic weights are plotted vertically to a scale in Fig. 2(*f*), a pole O is selected, and lines parallel to the resulting rays are drawn in Fig. 2(*g*). The funicular polygon (Fig. 2(*g*)), to a certain scale, represents the deflected beam subjected to the assumed loading. In order to obtain actual deflections the vertical distances in inches are multiplied by a scale factor. This scale factor is obtained by multiplying the scale of the horizontal beam distances by the scale of the vertical measurements in the force polygon, by the pole distance in inches, and, in the present example, also by dividing by the elastic modulus of the beam. Thus, the factor in the original solution was $\frac{5 \times 200 \times 4}{432{,}000} = 0.00925$ ft per in. Several deflection ordinates from a straight line passing through the end points of the beam are recorded in Fig. 2(*g*). The pole distance may be adjusted to make the scale factor more convenient.

The foundation modulus is $k_o = 300$ kips per cu ft. The deflections, in feet (Fig. 2(*g*)), correspond to the pressures (Fig. 2(*h*)) which are obtained by the relation $k_o\, y$, provided that, for awhile, the ends are assumed to be precisely at the surface. Then the beam is "locked" with the curvature shown and the entire curve is depressed uniformly into the foundation to obtain a static balance. Summing the foundation pressures already developed when the ends A and B are at the surface, $P = 2 \times \frac{1}{2} \times 5\ (2 \times 2.28 + 2 \times 4.17 + 4.89) = 89$ kips. Hence, $150 - 89 = 61$ kips remains to be uniformly distributed; that is, $s = \frac{61}{30} = 2.03$ kips per sq ft. Adding this value to the ordinates of Fig. 2(*h*) the results are as shown in Fig. 2(*i*). Averaging this with the initially assumed uniform pressure distribution of 5 kips per sq ft yields the results of one cycle of the semi-graphical solution. This is shown in Fig. 2(*j*).

To improve the accuracy of the results, another cycle may be made. Soil pressure loading shown in Fig. 2(*j*), together with the actual loads, are used in the next approximation. The $\frac{M}{I}$-diagram is shown in Fig. 2(*k*), and the elastic curve, in Fig. 2(*l*); the accompanying force polygon is not shown. After adjusting the results for static compatibility that accommodate the curve (Fig. 2(*l*)), the results shown in Fig. 2(*m*) are obtained. Averaging these results with those in Fig. 2(*j*) gives the results of the second cycle (Fig. 2(*n*)).

TABLE 1.—REFINEMENT POSSIBLE BY EXTRA CYCLES, EXAMPLE 1 (PRESSURE INTENSITY IN KIPS PER SQUARE FOOT)

| Point | Rigid beam | First cycle | Second cycle | Third cycle |
|---|---|---|---|---|
| 1 | 5.0 | 3.5 | 3.0 | 2.7 |
| 2 | 5.0 | 4.7 | 4.6 | 4.5 |
| 3 | 5.0 | 5.6 | 5.8 | 5.9 |
| 4 | 5.0 | 6.0 | 6.3 | 6.5 |

Two cycles are believed to give results that are sufficiently accurate for practical purposes. If more accuracy is desired, larger diagrams and more segments must be used, and, perhaps, a few more cycles of work must be performed. In a comparison of the results, the foregoing two cycles and one additional cycle are shown in Fig. 2(*o*) and Table 1.

## EXAMPLE 2

A "weightless" beam, 10 in. by 8 in., with the loading shown in Fig. 3(*a*) is resting on an elastic foundation. The modulus of the foundation $k_o = 200$ lb per cu in. The elastic modulus of the beam $E = 1.5 \times 10^6$ lb per sq in. (psi), and its moment of inertia $I = 426.7$ in.$^4$ Pressure distribution under this beam is determined at this point. (Mr. Hetényi has solved this problem mathematically.[14] Data for Example 2 were selected from his work.)

The resultant of the applied loads is found to be located 7.5 in. off center as shown in Fig. 3(*b*). This indicates that, in addition to the uniform settlement, angular displacement takes place. The contact area of the beam with the foundation is 1,200 sq in. and the moment of inertia of this area $I = \frac{1}{12} \times 10 \times 120^3 = 1{,}440{,}000$ in.$^4$ Hence, it is concluded that, if the beam were rigid, the pressure at end A would be $-11.23$ lb per sq in., and at end B, $-5.11$ lb per sq in., since $s_{A \text{ or } B} = \frac{P}{A} \pm \frac{M\,c}{I} = -\frac{9{,}800}{1{,}200} \mp \frac{9{,}800 \times 7.5 \times 60}{1{,}440{,}000} = -8.17 \mp 3.06$. This pressure variation is linear as shown in Fig. 3(*c*). As usual, minus signs indicate compression.

With actual applied loads as in Fig. 3(*a*) and pressure distribution as in Fig. 3(*c*), the bending moment diagram was calculated. It is approximated with the series of straight lines shown in Fig. 3(*d*). After computing the elastic weights, plotting them to a suitable scale, and selecting a pole (just as in the previous example), the elastic line shown in Fig. 3(*e*) is obtained. The deflections from the line passing through the ends are noted. Lack of symmetry in this case required that the entire curve be drawn in. The scale factor had

[14] "Beams on Elastic Foundation," by Miklós Hetényi, The Univ. of Michigan Press, Ann Arbor, Mich., 1946, p. 47.

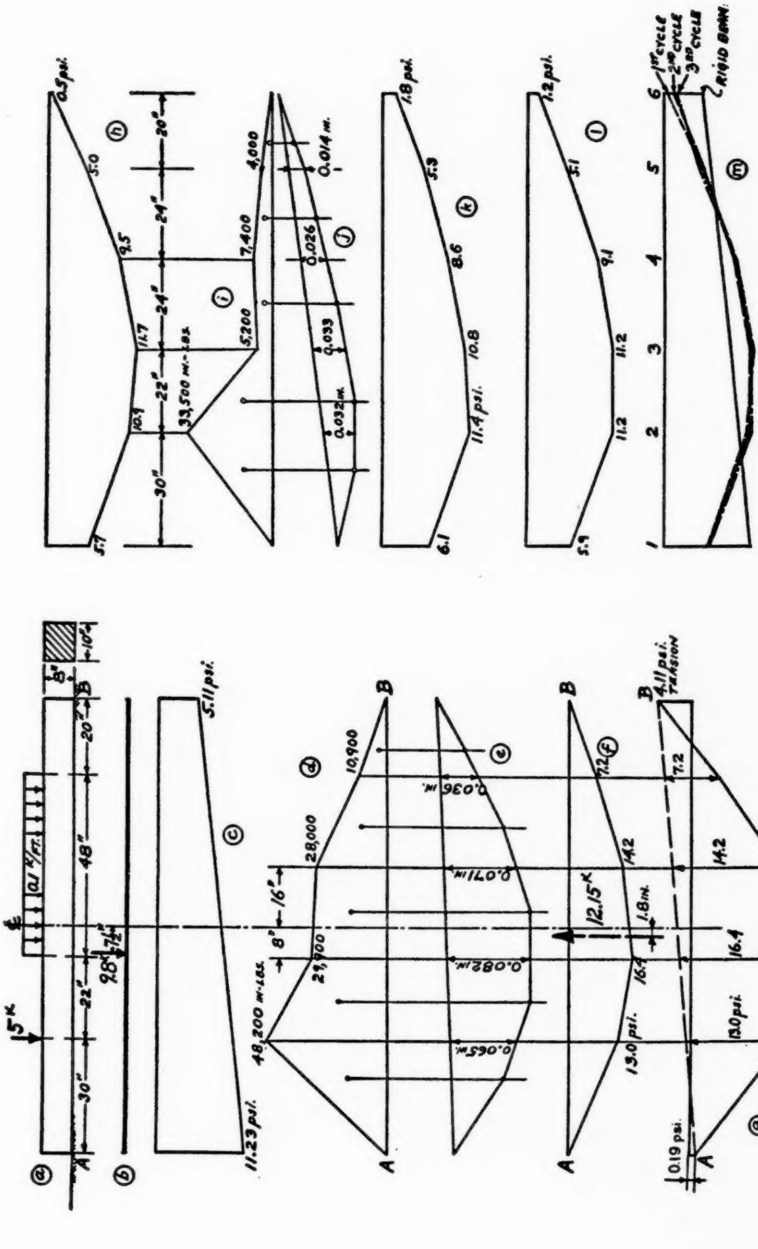

FIG. 3.—EXAMPLE 2, UNSYMMETRICAL LOADING

to be divided by the moment of inertia of the beam, as bending moment rather than the $\frac{M}{I}$ -diagram was used.

By "locking" the beam in the deflected shape just found, ends A and B are moved so that they are at the surface. The pressures developed in the foundation by the curved beam in this position are $k_o y$. These are shown in Fig. 3($f$). The resultant of these pressures is 12.15 kips ($12.15^k$) upward, 1.8 in. to the left of the center line. These quantities were obtained by subdividing the pressure area into triangles and rectangles and by taking their moments about one end of the beam. This resultant is in excess of the applied loads which total 918 kips; therefore, it is necessary to apply an upward pull of 2.35 kips to the "locked" curved beam. Moreover, the pressure resultant develops a clockwise resisting moment of only 21.8 kip-in., whereas, to keep the applied forces in balance, 73.5 kip-in. of moment is required. To develop the foundation resistance for the difference of 51.7 kip-in., an additional rotation to the curved beam (as a unit), of this magnitude, must be applied in the counterclockwise direction. Hence, from the known relations these pressure increments at the ends are: $s_A = \frac{P}{A} \pm \frac{M\,c}{I} = +\frac{2{,}350}{1{,}200} - \frac{51{,}700 \times 60}{1{,}440{,}000} = -0.19$ lb per sq in.; and, similarly, $s_B = -4.11$ lb per sq in. This pressure distribution varies linearly from point A to point B. Adding it algebraically to the pressure distribution shown in Fig. 3($f$), the distribution shown in Fig. 3($g$) is obtained. This diagram is statically compatible with the applied loads. Its resultant is equal and opposite to the 9.8-kip resultant of the applied loads.

TABLE 2.—REFINEMENT POSSIBLE BY EXTRA CYCLES, EXAMPLE 2 (PRESSURE INTENSITY IN POUNDS PER SQUARE INCH)

| Point | Rigid beam | First cycle | Second cycle | Third cycle | Exact |
|---|---|---|---|---|---|
| 1 | 11.23 | 5.7 | 5.9 | 5.9 | 6.07 |
| 2 | 9.70 | 10.9 | 11.2 | 11.3 | ... |
| 3 | 8.58 | 11.7 | 11.2 | 11.4 | ... |
| Center[a] | 8.17 | 11.0 | 10.5 | 10.7 | 10.39 |
| 4 | 7.36 | 9.5 | 9.1 | 9.2 | ... |
| 5 | 6.14 | 5.0 | 5.1 | 4.8 | ... |
| 6 | 5.11 | 0.5 | 1.2 | 1.3 | 1.26 |

[a] Pressures at the center are obtained by interpolation.

The average of the pressure distribution at the corresponding points in Figs. 3($c$) and 3($g$) yields the desired results for one cycle, as shown in Fig. 3($h$).

For the second cycle, the pressure distribution in Fig. 3($h$), together with the applied loads, gives an approximate moment diagram and the new elastic curve in Figs. 3($i$) and 3($j$), respectively. By "locking" this curved beam and adjusting the results for static compatibility, the pressure distribution shown in Fig. 3($k$) is obtained. The average of the values in Figs. 3($h$) and 3($k$) represents two cycles (see Fig. 3($l$)).

For comparison, the results of one additional cycle and the results of a mathematical solution,[14] are given in Table 2 and Fig. 3($m$). The agreement appears to be satisfactory.

## EXAMPLE 3

Consider a "weightless" beam, 8 in. by 6 in. (Fig. 4($a$)), with two equal loads of 100 lb each at the ends. The constants for the beam are: $E = 2.5 \times 10^6$ lb

per sq in. and $I = 144$ in.[4] The foundation modulus is assumed to vary linearly from $k_{oA} = 87.5$ lb per cu in. at end A to $k_{oB} = 12.5$ lb per cu in. at end B. Mathematical solutions for the deflection line and pressure distribution for these data are available.[15] Herein it is solved by the semi-graphical method.

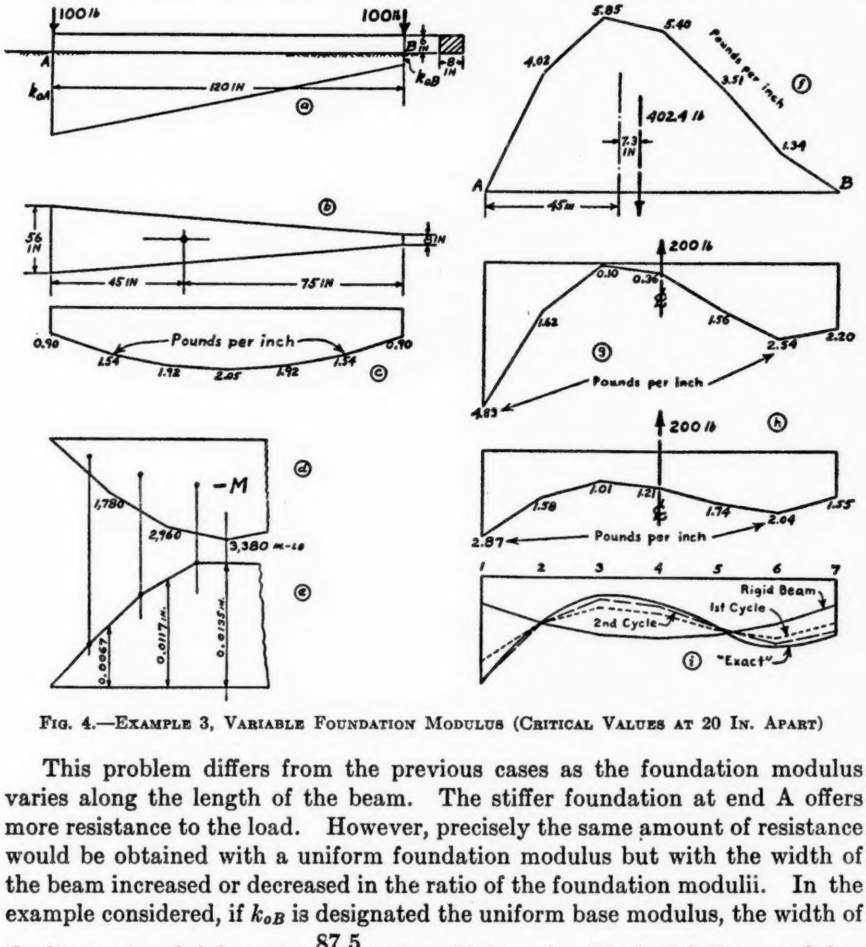


FIG. 4.—EXAMPLE 3, VARIABLE FOUNDATION MODULUS (CRITICAL VALUES AT 20 IN. APART)

This problem differs from the previous cases as the foundation modulus varies along the length of the beam. The stiffer foundation at end A offers more resistance to the load. However, precisely the same amount of resistance would be obtained with a uniform foundation modulus but with the width of the beam increased or decreased in the ratio of the foundation modulii. In the example considered, if $k_{oB}$ is designated the uniform base modulus, the width of the beam at end A becomes $\frac{87.5}{12.5} \times 8 = 56$ in. As the foundation modulus, in this case, varies linearly, the equivalent area over the region where $k_{oB}$ is the modulus is a trapezoid as shown in Fig. 4(*b*). Of course, any other modulus may be used as a base. The transformed area of the contact surface is 3,840 sq in.; its centroid is 45 in. from end A; and the moment of inertia of this area about its centroidal axis is $I = 3{,}744{,}000$ in.[4]

The loading is symmetrical in this case but the center of the foundation pressure resistance is not. The remaining parts of the problem are similar

[15] "Beams on Elastic Foundation," by Miklós Hetényi, The Univ. of Michigan Press, Ann Arbor, Mich., 1946, p. 111.

to Example 2. The pressure distribution for a rigid beam at point B is: $s_B = \frac{P}{A} \pm \frac{Mc}{I} = -\frac{200}{3{,}840} - \frac{200 \times 15 \times 75}{3{,}744{,}000} = -0.1122$ lb per sq in., or $8 \times s_B = -0.90$ lb per in. of beam. At the center line, $s_C = -\frac{200}{3{,}840} - \frac{200 \times 15 \times 15}{3{,}744{,}000} = -0.0641$ lb per sq in. on the fictitious beam, or $32 \times s_C = -2.05$ lb per in. on the actual rigid beam, 32 in. being the width of the transformed area at the center. In this manner pressure distribution on the rigid beam has been computed at 20-in. intervals. Results are plotted in Fig. 4(*c*). This pressure distribution, together with the two loads, yields the moment diagram shown in Fig. 4(*d*), and an elastic curve shown in Fig. 4(*e*). As before, placing ends A and B at the surface of the foundation, the pressure distribution caused by the deflected beam is as shown in Fig. 4(*f*). The deflections in Fig. 4(*f*) are multiplied by the corresponding variable foundation modulii. The resultant of this pressure, together with the two end loads, gives a total force of 602.4 lb acting 9.8 in. to the right of the centroidal axis. This is adjusted to be statically compatible, using the transformed contact area: $s_B = -\frac{602.4}{3{,}840} - \frac{602.4 \times 9.8 \times 75}{3{,}744{,}000} = -0.275$ lb per sq in., or $-8 \times 0.275 = -2.20$ lb per in. of actual beam, etc. These results are shown in Fig. 4(*g*). Averaging this curve with the initial distribution shown in Fig. 4(*c*), the results for one cycle are as shown in Fig. 4(*h*). Results for two cycles together with the analytical values[15] are given in Table 3 and Fig. 4(*i*). Again, the agreement found may be considered entirely satisfactory.

TABLE 3.—REFINEMENT POSSIBLE BY EXTRA CYCLES, EXAMPLE 3 (PRESSURE INTENSITY IN POUNDS PER INCH)

| Point | Rigid beam | First cycle | Second cycle | Exact |
|---|---|---|---|---|
| 1 | 0.90 | 2.87 | 3.54 | 3.62 |
| 2 | 1.54 | 1.58 | 1.57 | .. |
| 3 | 1.92 | 1.01 | 0.73 | 0.57 |
| 4 | 2.05 | 1.21 | 0.95 | .. |
| 5 | 1.92 | 1.74 | 1.70 | 1.65 |
| 6 | 1.54 | 2.04 | 2.22 | .. |
| 7 | 0.90 | 1.55 | 1.79 | 1.89 |

## CONCLUSIONS

The paper, with the illustrative examples, treats most of the important points that are encountered in practice in connection with beams on elastic foundation. Rather complex problems with variable moments of inertia of the beam, and variable foundation moduli can be solved with relative ease.

In the examples, the writer assumed consistently that the foundation can develop tension as well as compression stresses. This detail is of no importance during the performance of calculations. However, the end results must be interpreted with caution. If it is a soil problem, no tension is possible. The part of the beam that is experiencing only an upward pressure distribution must be statically compatible. Likewise, it must be borne in mind that, since examples involving soils were used, the foundation modulus is not a simple quantity. Reaction does not depend only on a deflection at a point; the adjoining areas also affect this value. To provide for this influence of adjoining areas, however, is one of the merits of the suggested procedure ($k_o$ may be com-

pensated in the light of this knowledge) and the problem remains comparatively simple.

With minor modifications, the method discussed is well adapted to other similar problems. Its utility for flexible floating structures is particularly notable.

The development of the paper followed the semi-graphical sequence; but the method is similar to the well known moment-area procedure, and, if desired, those who abhor graphics, may set up tables and perform all the necessary calculations accordingly. An attempt was made in this paper to illustrate the power and simplicity of the graphical method. All procedures are conventional. It should require little time to master the over-all scheme.

The method is not proposed for use in situations in which ready analytical solutions are available. Unfortunately, these cover only the simple cases.

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